Domain Theory Notes*

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Domain theory is the study of domains, which are sets of elements (B) with a partial ordering ($\sqsubseteq$). We therefore denote domains as $< B, \sqsubseteq >$. The partial ordering operator represents approximation and $a \sqsubseteq b$ should be read "a approximates b", though it is also common to read it as less than. The approximation comes from the idea that computations can be represented as streams of increasingly accurate approximations. This yields a natural way to represent infinite computations.

1 Domains and Partial Orders

The partial ordering operator must obey the following properties:

- **Reflexive** $x \sqsubseteq x$
- **Anti-symmetric** $x \sqsubseteq y \wedge y \sqsubseteq x \rightarrow x = y$
- **Transitive** $x \sqsubseteq y \wedge y \sqsubseteq z \rightarrow x \sqsubseteq z$

This operator can be depicted graphically using *Hasse Diagrams*, as shown in Figure 1. Nodes represent elements of the domain and edges represent the partial ordering. If there exists an edge from a to b then $a \sqsubseteq b$. Only edges which represent adjacent elements are explicitly drawn.

*These notes are based on the lecture which is based on Robert Cartwright and Rebecca Parson's notes on domain theory, *Domain Theory: An Introduction*

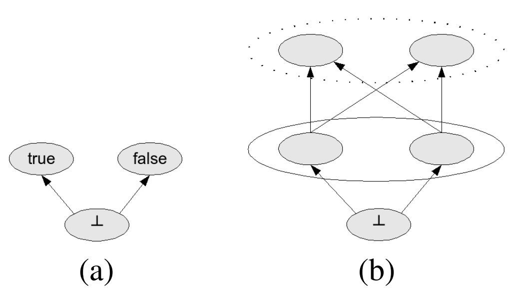
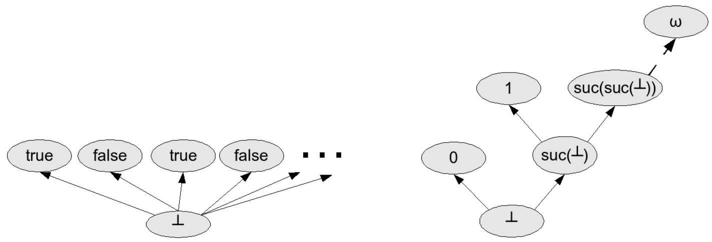


Figure 1: (a) A Hasse diagram of the booleans. (b) A simple non-finitary basis.



(a) (b)

Figure 2: Hasse domains for strict (a) and lazy (b) natural numbers.

Representing computations as streams of finite approximations leads to the need for an operator which determines what the final answer is approaching. We call this the *upper bound* of a domain. This can be written formally as

$$\sqcap S = x \rightarrow \forall y \in S.y \sqsubseteq x$$

An upper bound is a *least upper bound* if it approximates all upper bounds. Note that by reflexivity, all elements approximate themselves.

Domains with upper bounds are said to be *consistent*. The booleans, depicted in Figure 1a is an example of an inconsistent domain since true and false are not comparable.

2 Properties of Partial Orders

Subsets of a partial order are said to be *directed* if and only if ever *non-empty*, finite subset contains an upper bound. *Progressive* subsets are directed subsets which do not contain a maximum element and is said to be a *chain* if it is totally ordered, i.e. $\forall a, b.a \sqsubseteq b \vee b \sqsubseteq a$.

A domain is a *complete partial order* if every directed subset has a least upper bound in the original set. A *finitary basis* is a domain which is countable and every finite, consistent subset has a least upper bound. An example of a non-finitary basis is given in Figure 1b.

3 Programming Parallels

Following in the vein of representing computation as a stream of values, we considered the meaning of various domains. If a domain has a finite maximal element, then it represents a terminating computation since successive approximations will, in finite time, yield a value which does not contain $\perp$, the divergent computation.

In programming languages, most domains are flat. One such example domain is the strict integers, the Hasse diagram is given in Figure 2. A variation on this is the lazy natural numbers, also depicted in Figure 2. In this domain, abstract elements can be interpreted as lower bounds for their left-hand-side. For example $suc(\perp)$ can be understood as "at least 1". Solving for the fixed point in this domain $x = suc(x)$ yields the ω, drawn at the top-right.